On finite-density QCD at large N_c

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Abstract

Deryagin, Grigoriev, and Rubakov (DGR) have shown that in finite-density QCD at infinite $N_{\rm c}$ the Fermi surface is unstable with respect to the formation of chiral waves with wavenumber twice the Fermi momentum, while the BCS instability is suppressed. We show here that at large, but finite $N_{\rm c}$, the DGR instability only occurs in a finite window of chemical potentials from above $\Lambda_{\rm QCD}$ to $\mu_{\rm crit} \sim \exp(\gamma \ln^2 N_{\rm c} + O(\ln N_{\rm c} \ln\ln N_{\rm c}))\Lambda_{\rm QCD}$, where $\gamma \approx 0.02173$. Our analysis shows that, at least in the perturbative regime, the instability occurs only at extremely large $N_{\rm c}$, $N_{\rm c} \gtrsim 1000 N_{\rm f}$, where $N_{\rm f}$ is the number of flavors. We conclude that the DGR instability is not likely to occur in QCD with three colors, where the ground state at finite density is expected to be a color superconductor. We speculate on the possible structure of the ground state of finite-density QCD with very large $N_{\rm c}$.

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I. INTRODUCTION

In contrast with finite-temperature QCD, QCD at high baryonic densities remains remarkably poorly understood. One of the main reasons is the lack of lattice simulations due to the complex fermion determinant in finite-density QCD. Meanwhile, the physics in the core of the neutron stars, and possibly of heavy-ion collisions, depends crucially on the structure and properties of the ground state of QCD at finite densities.

It was suggested that at sufficiently high densities, the ground state of QCD is a color superconductor [1,2]. Such state arises from the instability of the Fermi surface under the formation of Cooper pairs of quarks. The superconducting phase of quark matter is the subject of many recent studies [3] and we will not discuss its properties in this paper. We will only note that a reliable treatment is currently available only in the perturbative regime of asymptotically high densities [4]; in the physically most interesting regime of moderate densities, QCD is strongly coupled and one has to resort to various toy models, e.g. those with four-fermion interactions [2].

To shed light on possible new phases that may occur in the non-perturbative regime of moderate baryonic densities, one might hope to be able to make use of alternative limits, such as the large $N_{\rm c}$ limit, where one takes the number of colors $N_{\rm c}$ to infinity, keeping $g^2N_{\rm c}$ fixed (g is the gauge coupling) [5]. This limit has proved to be a convenient framework for understanding many properties of QCD (for example, Zweig rule), although QCD at infinite $N_{\rm c}$ is still not analytically treatable. In the context of finite-density QCD, the first work that discussed the implications of the large $N_{\rm c}$ limit was done by Deryagin, Grigoriev, and Rubakov (DGR) [6]. DGR noticed that color superconductivity is suppressed at large $N_{\rm c}$ due to the fact that the Cooper pair is not a color singlet (the diagram responsible for color superconductivity is non-planar).[1] Working in the perturbative regime $g^2N_{\rm c} \ll 1$, DGR noticed another instability of the Fermi surface, this time with respect to the formation of chiral waves with wavenumber $2p_{\rm F}$, where $p_{\rm F}$ is the Fermi momentum. As shown by DGR, this instability is not suppressed in the limit $N_{\rm c} \to \infty$.

The purpose of this paper is to see what happens to the DGR instability at large but finite $N_{\rm c}$. Our motivation is to see whether the limit $N_{\rm c} \to \infty$ is relevant for the physics of high-density QCD at $N_{\rm c} = 3$. In this paper, we find that at any fixed value of the chemical potential μ, in order for the DGR instability to occur we require the number of colors $N_{\rm c}$ to be larger than some minimum value $N_{\rm c}(\mu)$, which grows with μ. What is surprising is that even for moderate values of μ, the minimum value $N_{\rm c}(\mu)$ is very large (of order of a few thousands for a modest chemical potential $\mu = 3\Lambda_{\rm QCD}$). Therefore one should not expect the large $N_{\rm c}$ limit to be of direct relevance for physics with $N_{\rm c} = 3$ at finite densities.

The paper is organized as follows. Section II reviews the results of DGR. A convenient technical approach to DGR instability which is based on renormalization group is developed in Sec. III and applied to the case of finite $N_{\rm c}$ in Sec. IV. Section V contains concluding remarks.

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[1] At arbitrary $N_{\rm c}$, using the technique of Ref. [4],the asymptotic behavior of the BCS gap can be found to be $\Delta \sim \mu \exp\left(-\sqrt{\frac{6N_{\rm c}}{N_{\rm c}+1}}\frac{\pi^2}{g}\right)$. This tends to 0 as $N_{\rm c} \to \infty$, provided one keeps $g^2N_{\rm c}$ fixed.

II. REVIEW OF DGR RESULTS

Let us review the key results of Ref. [6]. Throughout our paper, we assume all quarks are massless, and make no distinction between Fermi momentum and Fermi energy: $p_{\rm F} = \mu$. In the $N_{\rm c} \to \infty$ limit, the DGR result states that the Fermi surface is unstable under the development of chiral waves with wavenumber 2μ,

$$\langle \overline{\psi}(x)\psi(y)\rangle = e^{i\mathbf{P}\cdot(\mathbf{x}+\mathbf{y})}\int d^4q\, e^{-iq(x-y)} f(q) \tag{1}$$

where $\mathbf{P}$ is a vector with modulus $|\mathbf{P}| = \mu$ whose direction is fixed arbitrarily. Since $\overline{\psi}\psi$ is a color singlet, it survives the limit $N_{\rm c} \to \infty$.

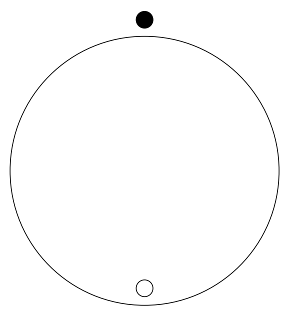

FIG. 1. The particle-hole pair

The condensate (1) can be interpreted as the formation of particle-hole pairs with total momentum $2\mathbf{P}$ (Fig. 1). In such a pair, both the particle and the hole are near the Fermi surface, and the momenta of the particle and the hole are both near $\mathbf{P}$. In this sense the condensate (1) is different from the usual chiral condensate $\langle\overline{\psi}\psi\rangle = \text{const}$, which corresponds to the pairing of a particle and a antiparticle moving in opposite directions. Moving in the same directions, the scattering between the particle and the hole is nearly in the foward direction, and since the amplitude of forward scattering is singular, one could expect the formation of the pair to be energetically favorable. In fact, this is the reason why the total momentum 2μ is special.

The function $f(q)$ has the physical meaning of the wave function of the pair in the center-of-mass frame, so $\mathbf{P}+\mathbf{q}$ is the momentum of the particle and $\mathbf{P}-\mathbf{q}$ is that of the hole. DGR found that the wave function is localized in an exponentially small region of momenta $q < \Delta_\perp$ where

$$\Delta_\perp \simeq \mu e^{-\pi/2h}, \qquad h^2 = \frac{g^2 N_{\rm c}}{4\pi^2}. \tag{2}$$

Recall that h is kept constant in the limit $N_{\rm c} \to \infty$. The binding energy of the pair is found to be at an even smaller scale,

$$E_{\rm bind} \sim \mu e^{-\pi/h}. \tag{3}$$

Both scales $\Delta_\perp$ and $E_{\rm bind}$ are parametrically larger than the non-perturbative scale $\Lambda_{\rm QCD} \sim \mu e^{-6/11h^2}$. For more details, see Ref. [6].

It may seem surprising that the DGR instability occurs in the perturbative regime. Indeed, the analogs of (1), in non-relativistic fermion systems, are the charge-density wave (CDW) and the spin-density wave (SDW). Since it is known that in three dimensions CDW and SDW do not develop at small four-fermion interaction, one could ask how such instability could occur at small $g^2 N_c$. The key observation is that in our case the effective four-fermion interaction is singular due to the $1/q^2$ behavior of the gluon propagator at small q. In general, this singularity is cut off by screening, but because the diagrams responsible for screening involve fermion loops, the screening effects at large N_c are of order $g^2\mu^2 \sim O(1/N_c)$ and therefore suppressed. This singular nature of the interaction explains why the DGR instability can occur perturbatively at large N_c.

The argument presented above also implies that at each value of the coupling h, there must be a lower limit on N_c, below which the interaction is not singular enough due to the screening, and the DGR instability disappears. This limit grows as one decreases h, or, equivalently, as the chemical potential increases. Finding this lower bound on N_c as a function of μ is the purpose of this paper.

III. RENORMALIZATION GROUP APPROACH TO DGR INSTABILITY

Before tackling our main problem, let us formulate an efficient RG technique that reproduces the results of DGR in the $N_c \to \infty$ limit. While in the limit $N_c \to \infty$ this technique does not give us anything new over what has been already found by DGR, it has the advantage that it can be applied to the case of finite N_c, where the effects of screening make the generalization of the original method of Ref. [6] very difficult, if at all possible. We will not try to rigorously justify the RG in this paper.

Let us stay in the Fermi liquid phase, where quarks are deconfined, and consider the scattering between a particle and a hole with momenta $\mathbf{P}+\mathbf{q}$ and $\mathbf{P}-\mathbf{q}$. The total momentum of the pair is 2μ. A singularity of this scattering amplitude in the upper half of the complex energy plane would signify an exponentially growing mode, i.e. an instability [7]. In terms of the diagrams, the most important contribution to the scattering amplitude comes from the ladder graphs (Fig. 2). Adding a rung to the ladder brings two more logarithms: one comes from the collinear divergence, i.e. the singular gluon propagator, and the other from the fact that the two new fermion propagators are near the mass shell. We will design the RG to resum these double logs.[2]

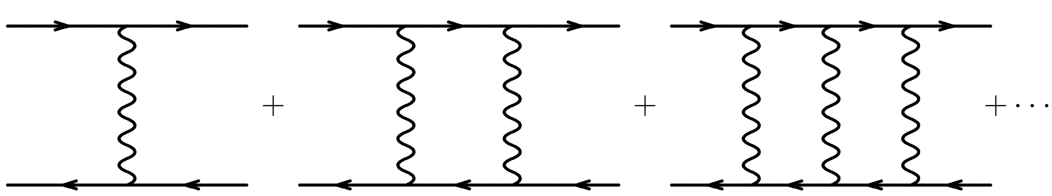

FIG. 2. Ladder approximation

[2] A similar but not identical RG procedure has been developed to resum the double logs in the BCS channel [4].

Our first step is to derive a 1+1 dimensional effective theory capable of describing the DGR instability. On the most naive level, such description exists due to the fact that the modes of interest move in directions close to $\pm\mathbf{P}$. Technically, the (1+1)D effective theory arises from integrating, in each Feynman graph, over the momentum components perpendicular to $\mathbf{P}$ [8].

Let us consider a ladder graph and ask what happens if one adds one more rung. The diagram now contains an extra loop integral,

$$\int \frac{d^4q}{(2\pi)^4}\, G(P+q)G(-P+q)D(q) \tag{4}$$

where G and D are fermion and gluon propagators respectively, and the Dirac structure of the fermion propagators is ignored for the purposes of the discussion presented below. Consider first the fermion line with momentum $\mathbf{P}+\mathbf{q}$. The component of the fermion momentum parallel to $\mathbf{P}$ will be denoted as $\mu + q_\parallel$, and those perpendicular to $\mathbf{P}$ will be denoted as $q_\perp$. Note that $q_\perp$ is a two-dimensional vector. We will assume that for all fermion lines in the Feynman diagram $q_\perp \sim \Delta$, where Δ is an arbitrary momentum scale much less than μ. In other words, we will be interested only in the modes located inside two small "patches" on the Fermi sphere, each having the size of order Δ in directions perpendicular to $\mathbf{P}$ (eventually, Δ will be identified with $\Delta_\perp$ in Eq. (2)). When $q_\parallel$ is also small compared to μ, the fermion propagator has the form

$$G(q) \sim \frac{1}{iq_0 + |\mathbf{P}+\mathbf{q}| - \mu} \approx \frac{1}{iq_0 + q_\parallel + \frac{q_\perp^2}{2\mu}}\,. \tag{5}$$

If $q_\parallel \gg q_\perp^2/\mu \sim \Delta^2/\mu$, the $q_\perp$ dependence drops out and the propagator is simply $(iq_0 + q_\parallel)^{-1}$. Therefore, in the regime $q_\parallel \gg \Delta^2/\mu$, the fermion propagator does not depend on the perpendicular (with respect to $\mathbf{P}$) momenta. In this regime, in Eq. (4) only the gluon propagator $D(q)$ depends on $q_\perp$. Hence, the integration over $q_\perp$ has the form

$$\int \frac{d^2q_\perp}{(2\pi)^2}\, \frac{1}{q_0^2 + q_\parallel^2 + q_\perp^2}\,. \tag{6}$$

If q_0 and $q_\parallel$ are not only small compared to μ, but also much smaller than Δ, then the integral over $q_\perp$ in Eq. (6) is a logarithmic one $\int d^2q_\perp/q_\perp^2$. The integral is cut off in the IR by $q_\parallel$ and in the UV by Δ and yields $\frac{g^2}{4\pi}\ln\frac{\Delta}{q_\parallel}$. Effectively, this integration replaces the internal gluon line by a four-fermion vertex $\frac{g^2}{4\pi}\ln\frac{\Delta}{q_\parallel}$, where $q_\parallel$ is determined by the momentum of the fermions coming in and out of the vertex (Fig. 3). Recall that the simplification takes place only in the region $\Delta \gg q_\parallel \gg \Delta^2/\mu$, as only in this region the integration over $q_\perp$ decouples from that over $q_\parallel$. At the end of this section we argue why the restriction of $q_\parallel$ to the region $\Delta^2/\mu \ll q_\parallel \ll \Delta$ is well justified.

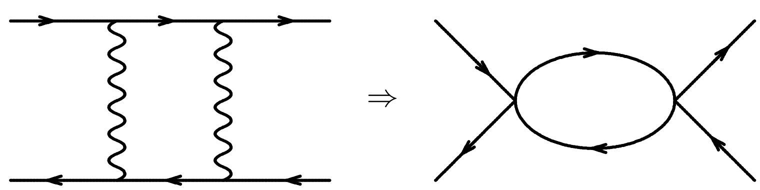

FIG. 3. Reducing one-gluon exchange to a point-like interaction in the effective theory

We have taken the integration over the perpendicular components of one particular gluon momentum, but nothing prevents us from integrating over the perpendicular components of *all* the gluon momenta. By doing this integration, we resum one set of logarithms (the one related to the collinear divergence) in a series of double logs. Now, as the only remaining integrals are over q_0 and $q_\parallel$, all Feynman diagrams are identical to those of some 1+1 dimensional model with a four-fermion interaction. Our task is to find out the precise form of the Lagrangian of this model.

First, we note that the kinetic term for the fermions in the effective theory can be obtained from the original Lagrangian by omitting spatial derivatives in directions other than z,

$$L_{\rm kin} = i\overline{\psi}\gamma^0\partial_0\psi + i\overline{\psi}\gamma^3\partial_3\psi + \mu\overline{\psi}\gamma^0\psi. \tag{7}$$

It is more convenient, however, to recast the Lagrangian (7) into the form of a (1+1)D theory of a doublet of Dirac fermions (which are two-component in (1+1)D) at *zero* chemical potential. This is indeed possible, since spinless fermions at finite chemical potential can be rewritten as one Dirac fermion at zero chemical potential (the modes near two points of the "Fermi surface" serve as its two components [9]). It is not surprising that in our case the spin-$\frac{1}{2}$ fermions can be rewritten as a doublet of (1+1)D Dirac fermions. Let us do it explicitly when $\mathbf{P}$ is directed along the z-axis, $\mathbf{P} = (0, 0, \mu)$. Denote the four components of the Dirac spinor ψ (in chiral basis) as $\psi^{\rm T} = (\psi_{\rm L1}, \psi_{\rm L2}, \psi_{\rm R1}, \psi_{\rm R2})$. The antiparticles have energy of order 2μ and decouple from the low-energy effective theory that is being derived. This allows us to consider only the components of ψ corresponding to particles, which are $\psi_{\rm L2}$ and $\psi_{\rm R1}$ when the particle's momentum is near $\mathbf{P}$, and $\psi_{\rm L1}$ and $\psi_{\rm R2}$ when it is near $-\mathbf{P}$. Although these fields are slowly varying in time, they still vary rapidly in space. To compensate for this spatial variation, we introduce new fields,

$$\varphi = \begin{pmatrix} e^{-i\mu z}\psi_{\rm L2} \\ e^{i\mu z}\psi_{\rm R2} \end{pmatrix}, \qquad \chi = \begin{pmatrix} e^{-i\mu z}\psi_{\rm R1} \\ e^{i\mu z}\psi_{\rm L1} \end{pmatrix} \tag{8}$$

which are soft in both space and time. We can now translate from the (3+1)D language of ψ to the (1+1)D language of φ and χ. The kinetic part of the Lagrangian (7) becomes

$$L_{\rm kin} = i\overline{\psi}\gamma^0\partial_0\psi + i\overline{\psi}\gamma^3\partial_3\psi + \mu\overline{\psi}\gamma^0\psi \to i\overline{\varphi}\gamma^\mu_{\rm 2D}\partial_\mu\varphi + i\overline{\chi}\gamma^\mu_{\rm 2D}\partial_\mu\chi. \tag{9}$$

What is the interaction term in the effective theory? A look at the Feynman diagram in Fig. 3 tells us that such interaction is of the current-current type. The current operator can also be translated into the (1+1)D counterparts,

$$\overline{\psi}\gamma^\mu\psi \to \overline{\varphi}\gamma^\mu_{\rm 2D}\varphi + \overline{\chi}\gamma^\mu_{\rm 2D}\chi \tag{10}$$

where $\gamma^\mu_{\rm 2D}$ are two (1+1)D Dirac matrices, $\gamma^0_{\rm 2D} = \sigma^1$, $\gamma^1_{\rm 2D} = -i\sigma^2$. Below we will write these matrices simply as γ^μ in all expressions belonging to the (1+1)D effective theory. Noting that each vertex in Fig. 3 corresponds to a factor of $\frac{g^2}{4\pi}\ln\frac{\Delta}{q_\parallel}$, where $q_\parallel$ is the parallel momentum transfer, we find that the Lagrangian of the (1+1)D effective theory is similar to that of the non-Abelian Thirring model

$$L_{\rm eff} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \frac{g^2}{4\pi}\ln\frac{\Delta}{q_\parallel}\left(\overline{\Psi}\gamma^\mu\frac{T^a}{2}\Psi\right)^2. \tag{11}$$

where we have combined the two fields φ and χ into a doublet Ψ. The only difference between (11) and the non-Abelian Thirring model is the dependence of the four-fermion coupling on the scale of the parallel momentum exchange $q_\parallel$. The theory (11) describes the interaction between fermions with perpendicular momenta of order Δ and parallel momenta between Δ^2/μ and Δ.

To understand the properties of the model (11), let us recall what is known about the conventional Thirring model, where the interaction term is $-\lambda(\overline{\Psi}\gamma^\mu\frac{T^a}{2}\Psi)^2$. The Thirring model is asymptotically free. The only diagram contributing to the β function at large $N_{\rm c}$ is the "zero-sound" diagram, Fig. 4.

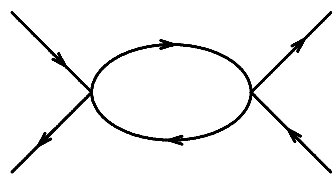

FIG. 4. The zero-sound diagram

The running of the coupling λ is governed by the RG equation,

$$\frac{\partial\lambda(s)}{\partial s} = \frac{N_{\rm c}}{\pi}\lambda^2(s)$$

where s is the RG parameter, and $\lambda(s)$ is the coupling at the energy scale Δe^{-s}. The coupling λ hits a Landau pole at $p \sim \Delta e^{-\pi/\lambda N_{\rm c}}$. The physics in the IR is characterized by the formation of the chiral condensate $\langle\overline{\Psi}\Psi\rangle$ which gives mass to the fermions. Using Eq. (8), one can see that $\langle\overline{\Psi}\Psi\rangle = \cos 2\mu z\langle\overline{\psi}\psi\rangle - i\sin 2\mu z\langle\overline{\psi}\gamma^0\gamma^3\psi\rangle$, so a constant $\langle\overline{\Psi}\Psi\rangle$ translates into space-dependent condensates $\langle\overline{\psi}\psi\rangle$ and $\langle\overline{\psi}\gamma^0\gamma^3\psi\rangle$.

These basic properties hold for the model (11) as well, but the estimation for the scale of the Landau pole is different. The latter can be found using RG. Now the RG equation needs to be written for a coupling which is a function of the parallel momentum transfer $q_\parallel$. At $s=0$,

$$\lambda(q_\parallel) = \frac{g^2}{4\pi}\ln\frac{\Delta}{q_\parallel}\,. \tag{12}$$

The RG equation is found from the diagram drawn in Fig. 4. The internal fermion lines have the momentum of order Δe^{-s}, which is much larger than the momentum of the external lines, therefore the momentum transfer at each vertex is Δe^{-s}. The RG equation, therefore, is

$$\frac{\partial}{\partial s}\lambda(s, q_\parallel) = \frac{N_{\rm c}}{\pi}\lambda^2(s, \Delta e^{-s}).$$

It is convenient to use the logarithmic parameter u, defined by $q_\parallel = \Delta e^{-u}$, and rewrite the RG equation as

$$\frac{\partial}{\partial s}\lambda(s, u) = \frac{N_{\rm c}}{\pi}\lambda^2(s, s). \tag{13}$$

The initial condition (12) becomes

$$\lambda(0, u) = \frac{g^2}{4\pi} u. \tag{14}$$

One should note that at the moment s of the RG evolution, all fermion modes with energy larger than Δe^{-s} have been integrated out, therefore the function $\lambda(s, u)$ is defined only for $u > s$. The solution to Eq. (13) with the initial condition Eq. (14) is

$$\lambda(s, u) = \frac{\pi}{N_c} f(s) + \frac{g^2}{4\pi}(u - s) \tag{15}$$

where $f(s)$ satisfies the equation

$$\frac{\partial}{\partial s} f(s) = h^2 + f^2(s) \tag{16}$$

and $h^2 = g^2 N_c / 4\pi^2$. Solving Eq. (16) one finds $f(s) = h \tan hs$, which hits a Landau pole at $s = s_L = \pi/2h$. The corresponding scale is $E_L = \Delta e^{-\pi/2h}$. Recall now that RG evolution occurs for $\Delta^2/\mu \ll q_\parallel \ll \Delta$. From this condition one finds that the Landau pole can only be achieved if $\Delta^2/\mu \lesssim E_L$ or $\Delta \lesssim \mu e^{-\pi/2h}$. Under this constraint, the maximal value of E_L is achieved when $\Delta \sim \mu e^{-\pi/2h}$, at which $E_L = \Delta^2/\mu \sim \mu e^{-\pi/h}$. Thus, the estimation for the Landau pole scale E_L and for Δ coincide with the result found by DGR for the binding energy of the particle-hole pair and the size of the pair wave function, Eqs. (2,3).

Now, it is easy to demonstrate why we were justified to consider only the region $\Delta^2/\mu \ll q_\parallel \ll \Delta$ in the argument presented above. On one hand, when $q_\parallel$ drops below the scale Δ^2/μ, we cannot neglect the dependence of the fermion propagator on $q_\perp$, which now acts as a cut off for the RG flow. Hence, for $q_\parallel \lesssim \Delta^2/\mu$, there is no RG flow in the effective (1+1)D theory and the Landau pole is never reached. On the other hand, when $q_\parallel$ becomes comparable with $q_\perp$ (i.e. Δ), we cannot neglect $q_\parallel$ dependence in the gluon propagator. One can estimate the effect of such dependence by noticing that the four-fermion coupling in the effective (1+1)D theory (11) now reads

$$\lambda(q_\parallel) = \frac{g^2}{8\pi} \ln\left(1 + \frac{\Delta^2}{q_\parallel^2}\right)$$

and the RG equation (16) becomes

$$\frac{\partial}{\partial s} f(s) = \frac{h^2}{1 + e^{-2s}} + f^2(s). \tag{17}$$

One can see that for $q_\parallel \gtrsim \Delta$, the RG flow in the effective (1+1)D is completely negligible. Therefore, to find the DGR instability we can restrict the values of $q_\parallel$ to lie between Δ^2/μ and Δ.

Having reproduced the DGR results by our RG procedure, let us turn to the case of large, but finite N_c.

IV. DGR INSTABILITY AT FINITE N_c

The RG technique described above can be very easily extended to the case of large but finite N_c. The effect of finite N_c is to cut off the IR singularity of the gluon propagator at small momentum exchange by Thomas-Fermi screening and Landau damping. The electric propagator becomes $(q^2+m^2)^{-1}$, and the magnetic propagator becomes $(q^2+im^2|q_0|/q)^{-1}$ [10], where m is the Thomas-Fermi screening scale of order $g\mu$. If the screening mass m is smaller than the scale of the Landau pole found in Sec. III, i.e. $\mu e^{-\pi/h}$, then our previous calculations are not affected. However, if $m>\mu e^{-\pi/h}$, we need to modify the RG to take into account the screening.

The screening affects the integration over perpendicular components of the gluon propagators: before these integrals were cut off by the parallel exchanged momentum $q_\parallel$, now it is cut off by the largest scale among $q_\parallel$ and m in the case of electric gluons, and among $q_\parallel$ and $m^{2/3}q_\parallel^{1/3}$ in the case of magnetic gluons. The effective (1+1)D theory is now a Thirring-like model with different scale-dependent couplings for the electric and magnetic interactions,

$$L_{\text{eff}} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - \lambda_0(q_\parallel)\left(\overline{\Psi}\gamma^0\frac{T^a}{2}\Psi\right)^2 + \lambda_1(q_\parallel)\left(\overline{\Psi}\gamma^1\frac{T^a}{2}\Psi\right)^2. \tag{18}$$

where

$$\begin{aligned}\lambda_0(q_\parallel) &= \frac{g^2}{4\pi}\ln\frac{\Delta}{\max(q_\parallel,m)}\\ \lambda_1(q_\parallel) &= \frac{g^2}{4\pi}\ln\frac{\Delta}{\max(q_\parallel,m^{2/3}q_\parallel^{1/3})}.\end{aligned}$$

The RG equations for $\lambda_+=(\lambda_0+\lambda_1)/2$ and $\lambda_-=(\lambda_0-\lambda_1)/2$ decouple:

$$\begin{aligned}\frac{\partial}{\partial s}\lambda_+(s,u) &= \frac{N_c}{\pi}\lambda_+^2(s,s)\\ \frac{\partial}{\partial s}\lambda_-(s,u) &= 0.\end{aligned} \tag{19}$$

where again $u=\ln\frac{\Delta}{q_\parallel}$. Therefore, only λ_+ changes during the RG evolution. The initial condition for λ_+ can be read from Eq. (18),

$$\lambda_+(0,u) = \begin{cases}\dfrac{g^2}{4\pi}u & \text{if } u<s_m\\ \dfrac{g^2}{4\pi}\left(\dfrac{5}{6}s_m+\dfrac{1}{6}u\right) & \text{if } u>s_m\end{cases} \tag{20}$$

where $s_m=\ln\frac{\Delta}{m}$. The solution to Eq. (19) with the initial condition (20) can be written in the form of Eq. (15), where $f(s)$ now satisfies the equation

$$\frac{\partial}{\partial s}f(s) = \begin{cases}f^2+h^2 & \text{if } s<s_m\\ f^2+\dfrac{h^2}{6} & \text{if } s>s_m\end{cases}.$$

The solution to this equation is

$$f(s) = \begin{cases} h \tan hs & \text{if } s < s_m \\ \dfrac{h}{\sqrt{6}} \tan \dfrac{h}{\sqrt{6}}(s+c) & \text{if } s > s_m \end{cases}$$

where c can be found by matching the solution at $s = s_m$:

$$c = \frac{\sqrt{6}}{h} \arctan(\sqrt{6} \tan hs_m) - s_m.$$

The Landau pole occurs at

$$s_{\mathrm{L}} = \frac{\sqrt{6}\pi}{2h} - c = \frac{\sqrt{6}}{h} \arctan(\frac{1}{\sqrt{6}} \cot hs_m) + s_m.$$

Recall that for the instability to really occur, the scale of the Landau pole should be larger than the scale Δ^2/μ, one finds a condition on m,

$$m = \Delta e^{-s_m} < \mu e^{-s_{\mathrm{L}} - s_m} = \mu \exp\Big[-\frac{\sqrt{6}}{h} \arctan\Big(\frac{1}{\sqrt{6}} \cot hs_m\Big) - 2s_m\Big].$$

One can maximize the right hand side (RHS) of this equation to find the maximum value of m where the Landau pole still can be achieved. One finds that for the Landau pole to be reached, m should be smaller than $m_{\max} = \mu e^{-c/h}$, where

$$c = \sqrt{6} \arctan \frac{1}{2} + 2 \arctan \sqrt{\frac{2}{3}} \approx 2.5051.$$

This restriction on m leads to a condition on N_{c} and μ for the DGR instability to occur. Recall that the Thomas-Fermi mass is

$$m = \sqrt{\frac{N_{\mathrm{f}}}{2\pi^2}} g\mu$$

(which is of order $N_{\mathrm{c}}{}^{-1/2}$), we see that at a fixed coupling $g^2 N_{\mathrm{c}}$ (or, equivalently, μ), there exists a lower bound on N_{c} where condition $m < \mu e^{-c/h}$ is satisfied. The lower bound can be easily found to be

$$N_{\mathrm{c}} \gtrsim 2N_{\mathrm{f}} h^2 e^{2c/h}. \tag{21}$$

Since our arguments rely on the comparison of scales, Eq. (21) contains an extra unknown coefficient of order 1 on the RHS. As the chemical potential μ increases, the effective coupling h decreases; using the one-loop beta function

$$h^2 = \frac{6}{11 \ln \frac{\mu}{\Lambda_{\mathrm{QCD}}}} \tag{22}$$

and according to Eq. (21) the lower bound on N_{c} increases. In reality, the numerical constant $2c$ in the exponent on the RHS of Eq. (21) is relatively large (≈ 5), so the lower bound is

already large at moderate values of μ. For example, if one uses the value of h corresponding to $\mu = 3\Lambda_{\rm QCD}$, the RHS of Eq. (21) is of order $1000N_{\rm f}$! Barring the possibility of a very small numerical constant on the RHS of Eq. (21), which seems unlikely, this lower bound is always much larger than 3.

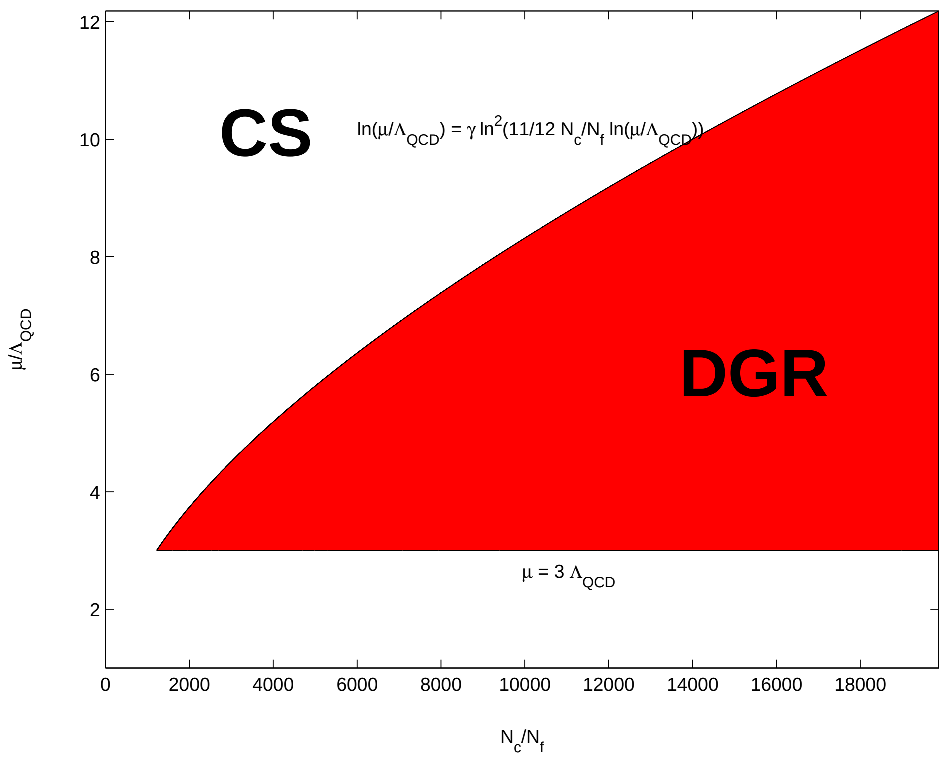

FIG. 5. Region of DGR instability in the $(N_{\rm c}, \mu)$ plane from Eq. (21). CS and DGR stand for regions with predominant color superconductivity and DGR instability, respectively.

From Eqs. (21,22), one can construct the phase diagram of QCD in the $(N_{\rm c}, \mu)$ plane. The result is shown in Fig. 5. In the shaded region, $N_{\rm c}$ satisfies the inequality (21), which means that DGR instability occurs. We restrict this region by the line $\mu = 3\Lambda_{\rm QCD}$, for below this line QCD is certainly strongly-coupled and not much can be said from our calculation. Above the curved line, inequality (21) is not satisfied, and the Fermi surface is stable in the DGR channel. However, the BCS instability is still there (though suppressed by large $N_{\rm c}$), thus implying that the ground state of QCD is a color superconductor in that region.

At any given (large) $N_{\rm c}$, the DGR instability occurs only in a finite window of the values of the chemical potential. The maximal value of μ where DGR instability still occurs, $\mu_{\rm crit}$, can be found by solving (21) with respect to μ. Asymptotically,

$$\mu_{\rm crit} \sim \exp(\gamma \ln^2 N_{\rm c} + O(\ln N_{\rm c} \ln\ln N_{\rm c}))\Lambda_{\rm QCD} \sim {N_{\rm c}}^{\gamma \ln N_{\rm c}} \Lambda_{\rm QCD}$$

where

$$\gamma = \frac{3}{22c^2} = 0.02173\ldots$$

The smallness of the numerical constant γ and the logarithmic dependence of μ_{crit} on N_{c} are the reasons why it requires a numerically large N_{c} for μ_{crit} to be as small as $3\Lambda_{\mathrm{QCD}}$. However, asymptotically μ_{crit} grows faster than any power of N_{c}.

V. CONCLUSION

In this paper we have seen that in finite-density QCD the Fermi surface is unstable under the DGR instability in a finite range of chemical potential. We have also found that the number of colors N_{c} needs to be numerically very large for the DGR instability to occur in perturbation theory. This indicates that at low N_{c} (like $N_{\mathrm{c}} = 3$), the DGR instability might not have a chance to realize itself at any value of the chemical potential and the only instability of the Fermi surface is the BCS one, which leads to color superconductivity.

Returning to the case of very large N_{c}, the next logical step is to ask what is the ground state once the Fermi liquid is unstable under the DGR particle-hole pairing. This seems to be a purely academic exercise due to the large N_{c} required, but it might still be interesting because of the possibility, at least in principle, of a new phase, distinct from the Fermi liquid and BCS superconducting phases in 3D fermionic systems. In the original paper [6], DGR constructed a "standing chiral wave state", in which $\langle\overline{\psi}\psi\rangle$ varies periodically in space with wavenumber 2μ. This state is periodic only along one spatial direction and does not break translational symmetry along the other two directions. Since translational symmetry cannot be broken in only one direction, such state cannot be the ground state of QCD.

One notices that a chiral wave with a particular wavevector utilizes only fermion modes in a small region with size $\Delta_\perp$ (Eq. (2)) near two opposite points on the Fermi sphere. It is clear how to make a state with energy smaller than the original DGR standing wave state. Indeed, one can pair up particles and holes in different pairs of opposite patches on the Fermi sphere. Since the size of each patch is exponentially small compared to the total area of the Fermi surface, one can have a large number of patches that do not overlap with each other. From the size of the patches one deduces that one can place a maximum of $e^{-\pi/h}$ patches on the sphere. The condensate has the form of a linear combination of $e^{i\mathbf{k}_i\cdot\mathbf{x}}$, where all $\mathbf{k}_i$ have modulus equal to 2μ but point in different directions. It is easy to estimate the energy gain from forming such a state. Indeed, the pairing affects fermions in a thin shell near the Fermi surface; the thickness of the shell is the scale at which we have found the Landau pole, i.e. $\mu e^{-\pi/h}$. Therefore, the fraction of fermions affected is $e^{-\pi/h}$, and each pair lowers the energy by $\mu e^{-\pi/h}$. Therefore, the gain in energy density is

$$\mu^4 e^{-2\pi/h}. \tag{23}$$

For comparison, the DGR standing wave state has the energy gain $\mu^4 e^{-3\pi/h}$. The factor of $e^{-\pi/h}$ difference is explained by the fact that DGR state involves only two patches on the Fermi surface with a relative area of $e^{-\pi/h}$.

Alternatively, it might be energetically more favorable for the patches on the Fermi sphere to be overlapping. In this case, a given particle (or hole) near the Fermi sphere participates in many pairings simultaneously. It could be expected that the binding energy of each individual pair is lower than the value it would have in the non-overlapping case, but nothing can be said about the total energy of the system. Indeed, our preliminary estimation

shows that the energy gain is still parametrically given by Eq. (23). Further investigation is required to find the true ground state of QCD at very large N_{c}.

Finally, let us note an interesting possibility that the ground state of finite-density QCD at very large N_{c} might be similar to the "tomographic Luttinger liquid" in 2D, advocated by Anderson as the normal state of high-T_c cuprates [11]. Such similarity could stem from the singular interaction between fermions moving in the same directions, which is also characteristic of tomographic Luttinger liquids. As in the case of the latter, one could expect the chiral symmetry to be unbroken, but the chiral response to be singular at wavenumber 2μ.

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